Retraction obstruction to time-varying stabilization

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Abstract

This paper address the problem of the global stabilization on a total space of a fiber bundle with a compact base space. We prove that, under mild assumptions (existence of a continuous section and forward unicity of solutions), no equilibrium of a continuous system defined on such a state space can be globally asymptotically uniformly stabilized using continuous time-varying feedback.

Key words: Retraction obstruction, Lyapunov stability, stabilization.

1 Introduction

Topological obstruction to stabilization is a long standing problem of control theory detailed in the introduction of [1]. There are two main topological obstructions to continuous stabilization. First, the *Brockett obstruction* is a local obstruction related to the structure of the controlled systems involving the nature of feedback controls [2]. Then, the *retraction obstruction* is a global obstruction related to the structure of the underlying state space: if the state space of the system has the structure of a vector bundle over a compact manifold, no continuous *static* feedback can globally stabilize an equilibrium. This result has been proved and its consequences studied in detail in [3].

The Brockett condition, which is necessary in the case of continuous time-invariant feedback controls, does not remain necessary for driftless controllable systems with time-varying feedback. The existence of such feedbacks has been proved in [4], while [5] gave an explicit design under an additional condition on the Control Lie Algebra (see [5, Assumption 1]). A natural question is, hence, to wonder whether continuous time-varying feedback controls could also avoid the retraction obstruction as suggested in the introduction of [6].

In [3, Remark 1], the authors mention that their result also handle the case of dynamic feedback. Indeed a dynamic feedback is usually seen as a dynamic extension where the augmented state is stable. Estimation of parameters or observer-based control are in this scope. In that case, the result of [3] is applicable directly, with the method exposed in their remark. Nevertheless, a time-varying static feedback is also a dynamic extension using a timer $\dot{\tau} = 1$. However in this situation, there is no convergence to a single equilibrium point, but to a submanifold; the result of [3] is therefore not applicable in this context, following [3, Remark 1].

The aim of this paper is to prove that, in the second case, the obstruction still remains: a time-varying feedback control which is globally asymptotically uniformly stabilizing a system defined on a fiber bundle with a compact manifold as its base space does not exist.

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2 Retraction obstruction

By a *manifold* we mean a smooth, positive dimensional, connected manifold without boundary. The definition of a *fiber bundle* is given for instance in [7].

Our purpose is to link the topological property of contractibility of a manifold to the existence of a globally asymptotically stable equilibrium. Let us introduce the definitions we will be using and some useful properties.

Definition 1 *Let E be a topological space and $x_0 \in E$. A* retraction *of E on x_0 is a continuous mapping $h : [0,1] \times E \to E$ such that for all $x \in E$, $h(0,x) = x$ and $h(1,x) = x_0$. A topological space E is said to be* contractible *if there exists a retraction of E.*

Proposition 2 *[8, Section 2.4] No compact manifold is contractible.*

Consider a controlled system defined by

$$\dot{y} = f_0(y,u) \qquad y \in \mathcal{N}, u \in \mathsf{U}, \tag{1}$$

with $\mathcal{N}$ a manifold and U a set of admissible controls, and where f_0 is a continuous vector field.

We wonder about the global stabilizability of the system via a time-varying feedback $u(y,t)$. To do that, we will set $\dot{\tau} = 1$ and look at the partial asymptotic stability of the closed-loop system:

$$\begin{aligned} \dot{y} &= f_0(y, u(y,\tau)) \\ \dot{\tau} &= 1 \end{aligned} . \tag{2}$$

Hence, let us introduce the definitions of partial stability we will be using. Those definitions are adapted from [9].

Definition 3 *Let $\mathcal{M} = \mathcal{N} \times \mathcal{T}$ be a product manifold. Consider $f = (f_1, f_2)$ a forward complete continuous vector field on $\mathcal{M}$ with the property of unicity of solutions in forward time. We denote by Φ the semiflow of f and p_1 the canonical projection on $\mathcal{N}$.*

(1) We say that $y_\infty \in \mathcal{N}$ is a partial equilibrium *if for all $\tau \in \mathcal{T}$, we have $f_1(y_\infty, \tau) = 0$.*
(2) A partial equilibrium $y_\infty \in \mathcal{N}$ is said to be partially stable uniformly in τ *if for all $U \subset \mathcal{N}$ neighborhood of y_∞ there exists $V \subset \mathcal{N}$ a neighborhood of y_∞ such that for all $y \in V$ and for all $\tau \in \mathcal{T}$, $p_1 \circ \Phi(t,(y,\tau)) \in U$ for all $t \geq 0$.*
(3) A partial equilibrium $y_\infty \in \mathcal{N}$ is said to be partially globally asymptotically stable uniformly in τ *if it is partially stable uniformly in τ and if for all $(y,\tau) \in \mathcal{M}$ we have $p_1 \circ \Phi(t,(y,\tau)) \to y_\infty$ when $t \to +\infty$.*

Remark 4 *The last item of definition 3 is slightly different from the more standard ones. Indeed, to prove our result, we only need the stability to be uniform with respect to τ. The uniformity with respect to τ of the convergence, which is usually required, is not necessary here.*

The following definition, inspired by [10, Chapter 12] about time-varying stabilizability, is here given in the partial stability context.

Definition 5 *The system (1) is said to be* globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback *if there exist a point $y_\infty \in \mathcal{N}$, a manifold $\mathcal{T}$, a continuous mapping $f_2 : \mathcal{N} \times \mathcal{T} \to \mathsf{T}\mathcal{T}$ with $f_2(y,\tau) \in \mathsf{T}_\tau\mathcal{T}$ for all $y \in \mathcal{N}$ and all $\tau \in \mathcal{T}$ and a continuous control law $u(y,\tau)$ such that y_∞ is a partial equilibrium of the closed loop system $(\dot{y}, \dot{\tau}) = (f_0(y, u(y,\tau)), f_2(y,\tau))$ and is partially globally asymptotically stable uniformly in τ.*

Remark 6 *Let us note that, taking $\mathcal{T} = \mathbb{R}$ and $f_2(y,\tau) = 1$, this definition boils down to the definition of global asymptotic stabilization by means of a continuous time-varying feedback. In the generalized time-varying setting, the variable τ can be stable or not, scalar or vector, bounded or not.*

In [3, Theorem 1], the authors use Proposition 2 to prove that if a manifold $\mathcal{N}$ admits a structure of fiber bundle over a compact manifold, then no continuous vector field over $\mathcal{N}$ can have an unique globally asymptotically stable equilibrium. Hence, they conclude that the system (1) cannot be globally asymptotically stabilized by means of a state feedback. Let us prove now that [3, Theorem 1] can be extended in the following way to the time-varying setting.

Theorem 7 *Assume that $\mathcal{N}$ is a manifold with a structure of fiber bundle over a compact manifold $\mathcal{Q}$. If there exists a continuous section of the bundle, then the system (1) is not globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback in such a way that the augmented vector field has the forward unicity of solutions property.*

Proof. *Ad absurdum*, assume that there exists a continuous dynamic feedback which globally asymptotically uniformly stabilizes the system (1) in such a way that the augmented vector field has the forward unicity of solutions. Let us denote by τ the added variable, and $\tau \in \mathcal{T}$. We have $\dot{\tau} = f_2(y,\tau)$, and the system (1) can be rewritten in an extended form, with $f_1(y,\tau) = f_0(y, u(y,\tau))$:

$$\begin{pmatrix} \dot{y} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} f_1(y,\tau) \\ f_2(y,\tau) \end{pmatrix}. \tag{3}$$

The equation (3) defines a continuous vector field f on the manifold $\mathcal{M} = \mathcal{N} \times \mathcal{T}$, with the forward unicity

of solutions property. Moreover, there exists a partially globally asymptotically stable equilibrium uniformly in τ denoted by $y_\infty \in \mathcal{N}$.
Let us denote $\pi_0 : \mathcal{N} \to \mathcal{Q}$ the fiber bundle projection. Set $p_1 : \mathcal{M} \to \mathcal{N}$ the first canonical projection and set $\pi = \pi_0 \circ p_1$. We denote $q_\infty = \pi_0(y_\infty)$. Similarly, fix τ_0 in $\mathcal{T}$ and set $\sigma(q) = (\sigma_0(q), \tau_0)$ where $\sigma_0 : \mathcal{Q} \to \mathcal{N}$ is a continuous section of π_0. Clearly, σ is a continuous section of π. We also note that the manifold $\mathcal{M}$ trivially inherits a structure of fiber bundle over $\mathcal{Q}$ with projection π. The vector field f is continuous and has the forward unicity of solutions property. Therefore, it admits a semiflow Φ[11]. Let us denote:

$$\begin{aligned} h : [0,1] \times \mathcal{Q} &\to \mathcal{Q} \\ (\lambda, q) \quad &\mapsto \begin{cases} \pi \circ \Phi\left(\ln\left(\frac{1}{1-\lambda}\right), \sigma(q)\right) & \text{if } \lambda \neq 1 \\ q_\infty & \text{if } \lambda = 1 \end{cases} \end{aligned}$$

Since we clearly have $h(0,q) = q$ and $h(1,q) = q_\infty$, let us prove the continuity of h. This mapping is obviously continuous on $[0,1) \times \mathcal{Q}$.
Let us show the continuity at $(1,q)$ for $q \in \mathcal{Q}$. Let $(\lambda_n, q_n) \in [0,1) \times \mathcal{Q}$ be a sequence of points converging to $(1,q)$. We set

$$t_n = \ln\left(\frac{1}{1-\lambda_n}\right), \; x_n = \sigma(q_n), \; x = \sigma(q).$$

We have $t_n \to +\infty$ and, by continuity of the section, $x_n \to x$. Let $U \subset \mathcal{Q}$ be a neighborhood of q_∞ and $U_0 = \pi_0^{-1}(U) \subset \mathcal{N}$ the corresponding neighborhood of y_∞. By partial stability uniformly in τ, there exists $V_0 \subset \mathcal{N}$ a neighborhood of y_∞ such that for all $y \in V_0$, all $\tau \in \mathcal{T}$ and all $t \geq 0$, we have $p_1 \circ \Phi(t, (y, \tau)) \subset U_0$.
On the other hand, the partial attractivity of y_∞ means that $p_1 \circ \Phi(t, x) \to y_\infty$ when $t \to \infty$. Thus, there exists $T > 0$ such that $p_1 \circ \Phi(T, x) \in V_0$. By continuity, there exists $N_1 > 0$ such that for all $n > N_1$ we have $p_1 \circ \Phi(T, x_n) \in V_0$. Therefore, for all $t \geq T$, we have $p_1 \circ \Phi(t, x_n) \in U_0$. But $t_n \to +\infty$, so there exists $N_2 > 0$ such that for all $n > N_2$, $t_n > T$. Thus, for all $n > N = \max(N_1, N_2)$, we have $p_1 \circ \Phi(t_n, x_n) \in U_0$. Hence, for all $n > N$, $h(\lambda_n, q_n) = \pi_0 \circ p_1 \circ \Phi(t_n, x_n) \in U$. Since U is an arbitrary neighborhood of q_∞, the mapping h is continuous.
However, the mapping h defines a retraction of the compact manifold $\mathcal{Q}$ on q_∞, which leads to the expected contradiction thanks to Proposition 2.

■

Example 8 *Let us consider the following system, defined on the circle:*

$$\ddot{\theta} = u, \qquad \theta \in \mathbb{S}^1. \tag{4}$$

By using the angular velocity $\omega = \dot{\theta}$ we can rewrite the system as

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = u \end{cases}. \tag{5}$$

Here the state space is the tangent space of the circle, denoted by $\mathsf{T}\mathbb{S}^1$. The tangent space of a manifold has always a structure of vector bundle over that manifold, and therefore $\mathsf{T}\mathbb{S}^1$ has a structure of vector bundle over the compact manifold $\mathbb{S}^1$ with projection π_0 given by:

$$\pi_0(\theta, \omega) = \theta.$$

Moreover, being a vector bundle, the tangent bundle admits σ_0, the zero section, as a continuous section. One may wonder if it is possible to design a continuous feedback control $u(\theta, \omega, t)$ globally stabilizing a state $(\theta_0, 0)$ such that the closed-loop system has uniqueness of solution in forward time.

Since $\mathbb{S}^1$ is compact, from Theorem 7 the system (5) cannot be globally asymptotically uniformly stabilized.

Finally, taking into account Theorem 7 and [3, Theorem 1], we have the following result: consider the system (1) defined on a manifold with a structure of fiber bundle over a compact manifold. If there exists a continuous section of the bundle, then no continuous dynamic feedback can globally asymptotically uniformly stabilize the system in such a way that the closed loop system has the forward unicity of solutions.

3 Conclusion

This paper extends [3, Theorem 1] to the case of time-varying feedback control. We prove that under mild assumptions, no continuous time-varying feedback control can avoid the retraction obstruction; that is, no continuous time-varying feedback control can globally asymptotically uniformly stabilize an equilibrium on a state space which has a structure of fiber bundle over a compact manifold.

This topological obstruction on compact manifolds prevents us from having continuous globally asymptotically uniformly stabilizing feedback (neither static nor time-varying nor dynamic). Moreover it is proved in [12] that the obstruction still remains for discontinuous autonomous vector fields or differential inclusions. Finally the topological obstruction appears to be a strong constraint on stabilization, and few possibilities remain. First, since the uniformity property of the partial stability is indeed being used in our proof, the possibility of non-uniform global stabilizability still remains open. Second, hybrid feedbacks can be considered as suggested in [12]. Finally, other notions of solutions for discontinuous systems exist; some of them might not inherit the same obstruction to global stabilization.

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